Noname manuscript No.
(will be inserted by the editor)

A general theorem of existence of quasi absolutely minimal Lipschitz extensions

Matthew J. Hirn · **Erwan Le Gruyer**

the date of receipt and acceptance should be inserted later

Abstract In this paper we consider a wide class of generalized Lipschitz extension problems and the corresponding problem of finding absolutely minimal Lipschitz extensions. We prove that if a minimal Lipschitz extension exists, then under certain other mild conditions, a quasi absolutely minimal Lipschitz extension must exist as well. Here we use the qualifier "quasi" to indicate that the extending function in question nearly satisfies the conditions of being an absolutely minimal Lipschitz extension, up to several factors that can be made arbitrarily small.

Keywords Minimal Lipschitz extension; absolutely minimal function; differentiable function

Mathematics Subject Classification (2000) 54C20, 58C25, 46T20, 49

1 Introduction

In this paper we attempt to generalize Aronsson's result on absolutely minimal Lipschitz extensions for scalar valued functions to a more general setting that includes a wide class of functions. The main result is the existence of a "quasi-AMLE," which intuitively is a function that nearly satisfies the conditions of absolutely minimal Lipschitz extensions.

Let $E \subset \mathbb{R}^d$ and $f : E \to \mathbb{R}$ be Lipschitz continuous, so that

$$\mathrm{Lip}(f;E) \triangleq \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|} < \infty.$$

Matthew J. Hirn
Yale University, Department of Mathematics, P.O. Box 208283, New Haven, Connecticut 06520-8283
E-mail: matthew.hirn@yale.edu

Erwan Le Gruyer
Institut National des Sciences Appliquées & IRMAR, 20 Avenue des Buttes de coësmes, 35043 Rennes cedex, France
E-mail: Erwan.Le-Gruyer@insa-rennes.fr

The original Lipschitz extension problem then asks the following question: is it possible to extend f to a function $F : \mathbb{R}^d \to \mathbb{R}$ such that

$$F(x) = f(x), \text{ for all } x \in E,$$
$$\mathrm{Lip}(F;\mathbb{R}^d) = \mathrm{Lip}(f;E).$$

By the work of McShane [11] and Whitney [24] in 1934, it is known that such an F exists and that two extensions can be written explicitly:

$$\Psi(x) \triangleq \inf_{y\in E}(f(y) + \mathrm{Lip}(f;E)\|x-y\|), \tag{1}$$

$$\Lambda(x) \triangleq \sup_{y\in E}(f(y) - \mathrm{Lip}(f;E)\|x-y\|). \tag{2}$$

In fact, the two extensions Ψ and Λ are extremal, so that if F is an arbitrary minimal Lipschitz extension of f, then $\Lambda \leq F \leq \Psi$. Thus, unless $\Lambda \equiv \Psi$, the extension F is not unique, and so one can search for an extending function F that satisfies additional properties.

In a series of papers in the $1960's$ [2–4], Aronsson proposed the notion of an absolutely minimal Lipschitz extension (AMLE), which is essentially the "locally best" Lipschitz extension. His original motivation for the concept was in conjunction with the infinity Laplacian and infinity harmonic functions. We first define the property of absolute minimality independently of the notion of an extension. A function $u : D \to \mathbb{R}$, $D \subset \mathbb{R}^d$, is absolutely minimal if

$$\mathrm{Lip}(u;V) = \mathrm{Lip}(u;\partial V), \quad \text{for all open } V \subset\subset D, \tag{3}$$

where ∂V denotes the boundary of V, $V \subset\subset D$ means that $\overline{V}$ is compact in D, and $\overline{V}$ is the closure of V. A function U is an AMLE for $f : E \to \mathbb{R}$ if U is a Lipschitz extension of f, and furthermore, if it is also absolutely minimal on $\mathbb{R}^d \setminus E$. That is:

$$U(x) = f(x), \text{ for all } x \in E,$$
$$\mathrm{Lip}(U;\mathbb{R}^d) = \mathrm{Lip}(f;E),$$
$$\mathrm{Lip}(U;V) = \mathrm{Lip}(U;\partial V), \text{ for all open } V \subset\subset \mathbb{R}^d \setminus E.$$

The existence of an AMLE was proved by Aronsson, and in 1993 Jensen [7] proved that AMLEs are unique under certain conditions (see also [6, 1]).

Since the work of Aronsson, there has been much research devoted to the study of AMLEs and problems related to them. For a discussion on many of these ideas, including self contained proofs of existence and uniqueness, we refer the reader to [5].

There are, though, several variants to the classical Lipschitz extension problem. A general formulation is the following: let $(\mathbb{X}, d_{\mathbb{X}})$ and (Z, d_Z) be two metric spaces, and define the Lipschitz constant of a function $f : E \to Z$, $E \subset \mathbb{X}$, as:

$$\mathrm{Lip}(f;E) \triangleq \sup_{\substack{x,y\in E\\ x\neq y}} \frac{d_Z(f(x), f(y))}{d_{\mathbb{X}}(x,y)}.$$

Given a fixed pair of metric spaces $(\mathbb{X}, d_{\mathbb{X}})$ and (Z, d_Z), as well as an arbitrary function $f : E \to Z$ with $\mathrm{Lip}(f;E) < \infty$, one can ask if it is possible to extend f to a function $F : \mathbb{X} \to Z$ such that $\mathrm{Lip}(F;\mathbb{X}) = \mathrm{Lip}(f;E)$. This is known as the isometric Lipschitz extension problem (or property, if it is known for a pair of metric spaces). Generally speaking it does not hold, although certain special cases beyond $\mathbb{X} = \mathbb{R}^d$ and $Z = \mathbb{R}$ do exist. For example, one can take $(\mathbb{X}, d_{\mathbb{X}})$ to be an arbitrary metric space and $Z = \mathbb{R}$ (simply adapt (1) and (2)). Another, more powerful generalization comes from the work of Kirszbraun [10] (and later, independently by Valentine [21]). In his paper in 1934, he proved that if $\mathbb{X}$ and Z are arbitrary Hilbert spaces, then they have the isometric Lipschitz extension property. Further examples exist; a more thorough discussion of the isometric Lipschitz extension property can be found in [22].

For pairs of metric spaces with the isometric Lipschitz extension property, one can then try to generalize the notion of an AMLE. Given that an AMLE should locally be the best possible such extension, the appropriate generalization for arbitrary metric spaces is the following. Suppose we are given a function $f : E \to Z$ and a minimal Lipschitz extension $U : \mathbb{X} \to Z$ such that $\mathrm{Lip}(U;\mathbb{X}) = \mathrm{Lip}(f;E)$. The function U is an AMLE if for every open subset $V \subset\subset \mathbb{X} \setminus E$ and every Lipschitz mapping $\widetilde{U} : \mathbb{X} \to Z$ that coincides with U on $\mathbb{X} \setminus V$, we have

$$\mathrm{Lip}(U;V) \leq \mathrm{Lip}(\widetilde{U};V). \tag{4}$$

When $(\mathbb{X}, d_{\mathbb{X}})$ is path connected, (4) is equivalent to (3). When $(\mathbb{X}, d_{\mathbb{X}})$ is an arbitrary length space and $Z = \mathbb{R}$, there are several proofs of existence of AMLE's [13, 8, 19] (some under certain conditions). The proof of uniqueness in this scenario is given in [16].

Extending results on AMLE's to non scalar valued functions presents many difficulties, which in turn has limited the number of results along this avenue. Two recent papers have made significant progress, though. In [15], the authors consider the case when $(\mathbb{X}, d_{\mathbb{X}})$ is a locally compact length space, and (Z, d_Z) is a metric tree; they are able to prove existence and uniqueness of AMLE's for this pairing. The case of vector valued functions with $(\mathbb{X}, d_{\mathbb{X}}) = \mathbb{R}^d$ and $(Z, d_Z) = \mathbb{R}^m$ is considered in [18]. In this case an AMLE is not necessarily unique, so the authors propose a stronger condition called tightness for which they are able to get existence and uniqueness results in some cases.

In this paper we seek to add to the progress on the theory of non scalar valued AMLE's. We propose a generalized notion of an AMLE for a large class of isometric Lipschitz extension problems, and prove a general theorem for the existence of what we call a quasi-AMLE. A quasi-AMLE is, essentially, a minimal Lipschitz extension that comes within ε of satisfying (3). We work not only with general metric spaces, but also a general functional Φ that replaces the specific functional Lip. In our setting Lip is an example of the type of functionals we consider, but others exist as well. For example, one can also take α-Hölder functions in which $\Phi = \mathrm{Lip}_\alpha$, where

$$\mathrm{Lip}_\alpha(f;E) \triangleq \sup_{\substack{x,y \in E \\ x \neq y}} \frac{d_Z(f(x), f(y))}{d_{\mathbb{X}}(x,y)^\alpha}, \quad \alpha \in (0,1].$$

An isometric extension with $\Phi = \mathrm{Lip}_\alpha$ is possible for certain pairs of metric spaces and certain values of α (see, once again, [22]).

A completely different type of functional is given in [20]. If we consider the classic Lipschitz extension problem as the zero-order extension, then for the first order extension we would want an extension that minimizes $\mathrm{Lip}(\nabla F;\mathbb{R}^d)$. In this case, one is given a subset $E \subset \mathbb{R}^d$ and a 1-field $\mathscr{P}_E = \{P_x\}_{x\in E} \subset \mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$, consisting of first order polynomials mapping $\mathbb{R}^d$ to $\mathbb{R}$ that are indexed by the elements of E. The goal is to extend $\mathscr{P}_E$ to a function $F \in C^{1,1}(\mathbb{R}^d)$ such that two conditions are satisfied: 1.) for each $x \in E$, the first order Taylor polynomial J_xF of F at x agrees with P_x; and 2.) $\mathrm{Lip}(\nabla F;\mathbb{R}^d)$ is minimal. By a result of Le Gruyer [20], such an extension is guaranteed to exist with Lipschitz constant $\Gamma^1(\mathscr{P}_E)$, assuming that $\Gamma^1(\mathscr{P}_E) < \infty$ (here Γ^1 is a functional defined in [20]). The functional Γ^1 can be thought of as the Lipschitz constant for 1-fields. By the results of this paper, one is guaranteed the existence of a quasi-AMLE for this setting as well.

2 Setup and the main theorem

2.1 Metric spaces

Let $(\mathbb{X},d_\mathbb{X})$ and (Z,d_Z) be metric spaces. We will consider functions of the form $f: E \to Z$, where $E \subset \mathbb{X}$. For the range, we require:

1. (Z,d_Z) is a complete metric space.

For the domain, $(\mathbb{X},d_\mathbb{X})$, we require some additional geometrical properties:

1. $(\mathbb{X},d_\mathbb{X})$ is complete and proper.
2. $(\mathbb{X},d_\mathbb{X})$ is midpoint convex. Recall that this means that for any two points $x,y \in \mathbb{X}$, $x \neq y$, there exists a third point $m(x,y) \in \mathbb{X}$ for which
$$d_\mathbb{X}(x,m(x,y)) = d_\mathbb{X}(m(x,y),y) = \frac{1}{2}d_\mathbb{X}(x,y).$$
Such a point $m(x,y)$ is called the midpoint and $m: \mathbb{X}\times\mathbb{X} \to \mathbb{X}$ is called the midpoint map. Since we have also assumed that $(\mathbb{X},d_\mathbb{X})$ is complete, this implies that $(\mathbb{X},d_\mathbb{X})$ is a geodesic (or strongly intrinsic) metric space. By definition then, every two points $x,y \in \mathbb{X}$ are joined by a geodesic curve with finite length equal to $d_\mathbb{X}(x,y)$.
3. $(\mathbb{X},d_\mathbb{X})$ is distance convex, so that for all $x,y,z \in \mathbb{X}$, $x \neq y$,
$$d_\mathbb{X}(m(x,y),z) \leq \frac{1}{2}(d_\mathbb{X}(x,z) + d_\mathbb{X}(y,z)).$$
Note that this implies that $(\mathbb{X},d_\mathbb{X})$ is ball convex, which in turn implies that every ball in $(\mathbb{X},d_\mathbb{X})$ is totally convex. By definition, this means that for any two points x,y lying in a ball $B \subset \mathbb{X}$, the geodesic connecting them lies entirely in B. Ball convexity also implies that the midpoint map is unique, and, furthermore, since $(\mathbb{X},d_\mathbb{X})$ is also complete, that the geodesic between two points is unique.

We remark that the $(\mathbb{X},d_\mathbb{X})$ is path connected by these assumptions, and so (4) is equivalent to (3) for all of the cases that we consider here.

2.2 Notation

Set $\mathbb{N} \triangleq \{0,1,2,\dots\}$, $\mathbb{N}^* \triangleq \{1,2,3,\dots\}$, and $\mathbb{R}^+ \triangleq [0,\infty)$. Let S be an arbitrary subset of $\mathbb{X}$, i.e., $S \subset \mathbb{X}$, and let $\mathring{S}$ and $\overline{S}$ denote the interior of S and the closure of S, respectively. For any $x \in \mathbb{X}$ and $S \subset \mathbb{X}$, set

$$d_{\mathbb{X}}(x,S) \triangleq \inf\{d_{\mathbb{X}}(x,y) \mid y \in S\}.$$

For each $x \in \mathbb{X}$ and $r > 0$, let $B(x;r)$ denote the open ball of radius r centered at x:

$$B(x;r) \triangleq \{y \in \mathbb{X} \mid d_{\mathbb{X}}(x,y) < r\}.$$

We will often utilize a particular type of ball: for any $x,y \in \mathbb{X}$, define

$$B_{1/2}(x,y) \triangleq B\left(m(x,y); \frac{1}{2} d_{\mathbb{X}}(x,y)\right).$$

By $\mathscr{F}(\mathbb{X},Z)$, we denote the space of functions mapping subsets of $\mathbb{X}$ into Z:

$$\mathscr{F}(\mathbb{X},Z) \triangleq \{f : E \to Z \mid E \subset \mathbb{X}\}.$$

If $f \in \mathscr{F}(\mathbb{X},Z)$, set $\mathrm{dom}(f)$ to be the domain of f. We shall use $E = \mathrm{dom}(f)$ interchangeably depending on the situation. We also set $\mathscr{K}(\mathbb{X})$ to be the set of all compact subsets of $\mathbb{X}$.

2.3 General Lipschitz extensions

Definition 1 Given $f \in \mathscr{F}(\mathbb{X},Z)$, a function $F \in \mathscr{F}(\mathbb{X},Z)$ is an *extension* of f if

$$\mathrm{dom}(f) \subset \mathrm{dom}(F) \quad \text{and} \quad F(x) = f(x), \text{ for all } x \in \mathrm{dom}(f).$$

We shall be interested in arbitrary functionals Φ with domain $\mathscr{F}(\mathbb{X},Z)$ such that:

$$\begin{aligned} \Phi : \mathscr{F}(\mathbb{X},Z) &\to \mathscr{F}(\mathbb{X}\times\mathbb{X}, \mathbb{R}^+ \cup \{\infty\}) \\ f &\mapsto \Phi(f) : \mathrm{dom}(f) \times \mathrm{dom}(f) \to \mathbb{R}^+ \cup \{\infty\}. \end{aligned}$$

In order to simplify the notation slightly, for any $f \in \mathscr{F}(\mathbb{X},Z)$ and $x,y \in \mathrm{dom}(f)$, we set

$$\Phi(f;x,y) \triangleq \Phi(f)(x,y).$$

We also extend the map $\Phi(f)$ to subsets $D \subset \mathrm{dom}(f)$ as follows:

$$\Phi(f;D) \triangleq \sup_{\substack{x,y \in D \\ x \neq y}} \Phi(f;x,y).$$

The map Φ serves as a generalization of the standard Lipschitz constant $\mathrm{Lip}(f;D)$ first introduced in Section 1. As such, one can think of it in the context of minimal extensions. Let $\mathscr{F}_{\Phi}(\mathbb{X},Z) \subset \mathscr{F}(\mathbb{X},Z)$ denote those functions in $\mathscr{F}(\mathbb{X},Z)$ for which Φ is finite, i.e.,

$$\mathscr{F}_{\Phi}(\mathbb{X},Z) \triangleq \{f \in \mathscr{F}(\mathbb{X},Z) \mid \Phi(f;\mathrm{dom}(f)) < \infty\}.$$

We then have the following definition.

Definition 2 Let $f \in \mathscr{F}_\Phi(\mathbb{X}, Z)$ and let $F \in F_\Phi(\mathbb{X}, Z)$ be an extension of f. We say F is a *minimal extension* of the function f if

$$\Phi(F; \mathrm{dom}(F)) = \Phi(f; \mathrm{dom}(f)). \tag{5}$$

One can then generalize the notion of an absolutely minimal Lipschitz extension (AMLE) in the following way:

Definition 3 Let $f \in \mathscr{F}_\Phi(\mathbb{X}, Z)$ and let $U \in \mathscr{F}_\Phi(\mathbb{X}, Z)$ be a minimal extension of f. We say that U is an *absolutely minimal Lipschitz extension* of the function f if

$$\Phi(U; V) = \Phi(U; \partial V), \quad \text{for all open } V \subset\subset \mathbb{X} \setminus \mathrm{dom}(f). \tag{6}$$

Note, since $(\mathbb{X}, d_\mathbb{X})$ is implicitly assumed to be a geodesic metric space, the open set V cannot also be closed. Thus ∂V is nonempty.

In this paper we prove the existence of a function U that is a minimal extension of f, and that "nearly" satisfies (6). In order to make this statement precise, we first specify the properties that Φ must satisfy, and then formalize what we mean by "nearly." Before we get to either task, though, we first define the following family of curves.

Definition 4 For each $x, y \in \mathbb{X}$, $x \neq y$, let $\Gamma(x, y)$ denote the set of curves

$$\gamma : [0,1] \to B_{1/2}(x, y),$$

such that $\gamma(0) = x$, $\gamma(1) = y$, γ is continuous, and γ is monotone in the following sense:

$$\text{If } 0 \leq t_1 < t_2 \leq 1, \text{ then } d_\mathbb{X}(\gamma(0), \gamma(t_1)) < d_\mathbb{X}(\gamma(0), \gamma(t_2)).$$

The required properties of Φ are the following (note that (P_1) has already been stated as a definition):

(P_0) Φ is symmetric and nonnegative:
For all $f \in \mathscr{F}(\mathbb{X}, Z)$ and for all $x, y \in \mathrm{dom}(f)$,

$$\Phi(f; x, y) = \Phi(f; y, x) \geq 0.$$

(P_1) Pointwise evaluation:
For all $f \in \mathscr{F}(\mathbb{X}, Z)$ and for all $D \subset \mathrm{dom}(f)$,

$$\Phi(f; D) = \sup_{\substack{x, y \in D \\ x \neq y}} \Phi(f; x, y).$$

(P_2) Φ is minimal:
For all $f \in \mathscr{F}_\Phi(\mathbb{X}, Z)$ and for all $D \subset \mathbb{X}$ such that $\mathrm{dom}(f) \subset D$, there exists an extension $F : D \to Z$ of f such that

$$\Phi(F; D) = \Phi(f; \mathrm{dom}(f)).$$

(P_3) Chasles' inequality:
For all $f \in \mathscr{F}_\Phi(\mathbb{X},Z)$ and for all $x,y \in \mathrm{dom}(f)$, $x \neq y$, such that $B_{1/2}(x,y) \subset \mathrm{dom}(f)$, there exists a curve $\gamma \in \Gamma(x,y)$ such that

$$\Phi(f;x,y) \leq \inf_{t\in[0,1]} \max\{\Phi(f;x,\gamma(t)),\Phi(f;\gamma(t),y)\}.$$

(P_4) Continuity of Φ:
Let $f \in \mathscr{F}_\Phi(\mathbb{X},Z)$. For each $x,y \in \mathrm{dom}(f)$, $x \neq y$, and for all $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon, d_\mathbb{X}(x,y)) > 0$ such that

$$\text{For all } z \in B(y;\eta) \cap \mathrm{dom}(f), \quad |\Phi(f;x,y) - \Phi(f;x,z)| < \varepsilon.$$

(P_5) Continuity of f:
If $f \in \mathscr{F}_\Phi(\mathbb{X},Z)$, then $f : \mathrm{dom}(f) \to Z$ is a continuous function.

2.4 Examples of the metric spaces and the functional Φ

Before moving on, we give some examples of the metric spaces $(\mathbb{X},d_\mathbb{X})$ and (Z,d_Z) along with the functional Φ.

2.4.1 Scalar valued Lipschitz extensions

The scalar valued case discussed at the outset is of course one such example. Indeed, one can take $\mathbb{X} = \mathbb{R}^d$ and $d_\mathbb{X}(x,y) = \|x-y\|$, where $\|\cdot\|$ is the Euclidean distance. For the range, set $Z = \mathbb{R}$ and $d_Z(a,b) = |a-b|$. For any $f : E \to \mathbb{R}$, where $E \subset \mathbb{X}$, define Φ as:

$$\Phi(f;x,y) = \mathrm{Lip}(f;x,y) \triangleq \frac{|f(x)-f(y)|}{\|x-y\|},$$
$$\Phi(f;E) = \mathrm{Lip}(f;E) \triangleq \sup_{\substack{x,y\in E\\ x\neq y}} \mathrm{Lip}(f;x,y).$$

Clearly (P_0) and (P_1) are satisfied. By the work of McShane [11] and Whitney [24], (P_2) is also satisfied. (P_3) is satisfied with $\gamma(t) = (1-t)x+ty$, and (P_4) and (P_5) are easy to verify.

2.4.2 Lipschitz mappings between Hilbert spaces

More generally, one can take $(\mathbb{X},d_\mathbb{X}) = \mathscr{H}_1$ and $(Z,d_Z) = \mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces (note, since we assume that $(\mathbb{X},d_\mathbb{X})$ is proper, there are some restrictions on $\mathscr{H}_1$). Then for any $f : E \to \mathscr{H}_2$, with $E \subset \mathscr{H}_1$, define Φ as:

$$\Phi(f;x,y) = \mathrm{Lip}(f;x,y) \triangleq \frac{\|f(x)-f(y)\|_{\mathscr{H}_2}}{\|x-y\|_{\mathscr{H}_1}},$$
$$\Phi(f;E) = \mathrm{Lip}(f;E) \triangleq \sup_{\substack{x,y\in E\\ x\neq y}} \mathrm{Lip}(f;x,y).$$

Clearly (P_0) and (P_1) are satisfied. By the work of Kirszbraun [10] and later Valentine [21], (P_2) is also satisfied. (P_3) is satisfied with $\gamma(t) = (1-t)x + ty$, and (P_4) and (P_5) are easy to verify.

2.4.3 Lipschitz-Hölder mappings between metric spaces

More generally still, one can take any pair of metric spaces $(\mathbb{X}, d_{\mathbb{X}})$ and (Z, d_Z) satisfying the assumptions of Section 2.1. We can also define a slightly more general Lipschitz-Hölder constant with a parameter $0 < \alpha \leq 1$. In this case, for any $f : E \to Z$, $E \subset \mathbb{X}$, define $\Phi = \Phi_\alpha$ as:

$$\Phi_\alpha(f;x,y) = \mathrm{Lip}_\alpha(f;x,y) \triangleq \frac{d_Z(f(x),f(y))}{d_{\mathbb{X}}(x,y)^\alpha},$$
$$\Phi_\alpha(f;E) = \mathrm{Lip}_\alpha(f;E) \triangleq \sup_{\substack{x,y\in E\\ x\neq y}} \mathrm{Lip}_\alpha(f;x,y).$$

Clearly (P_0), (P_1), (P_4), and (P_5) are satisfied. For (P_3), we can take $\gamma \in \Gamma(x,y)$ to be the unique geodesic between x and y. All that remains to check, then, is (P_2), the existence of a minimal extension. Such a condition is not satisfied between two metric spaces in general, although special cases beyond those already mentioned do exist. We highlight the following examples, taken from [22]:

1. Take $(\mathbb{X}, d_{\mathbb{X}})$ to be any metric space, take $(Z, d_Z) = \ell_n^\infty$, and take $\alpha = 1$. Note, we set ℓ_n^∞ to denote $\mathbb{R}^n$ with the norm $\|x\|_\infty \triangleq \max\{|x_j| \mid j = 1,\ldots,n\}$. See [22], Theorem 11.2, Chapter 3, as well as the discussion afterwards. See also [14,9].
2. Let $\mathscr{H}$ be a Hilbert space, take $(\mathbb{X}, d_{\mathbb{X}}) = (Z, d_Z) = \mathscr{H}$, and let $\alpha \in (0,1]$. See [22], Theorem 11.3, Chapter 3, as well as [12,17].
3. Take $(\mathbb{X}, d_{\mathbb{X}})$ to be any metric space and let $(Z, d_Z) = L^p(\mathscr{N}, \nu)$, where $(\mathscr{N}, \nu)$ is a σ-finite measure space and $p \in [1,\infty]$. Set $p' \triangleq p/(p-1)$. Using [22], Theorem 19.1, Chapter 5, we have that property (P_2) holds when:
   (a) $2 \leq p < \infty$ and $0 < \alpha \leq 1/p$.
   (b) $1 < p \leq 2$ and $0 < \alpha \leq 1/p'$.

2.4.4 1*-fields*

Let $(\mathbb{X}, d_{\mathbb{X}}) = \mathbb{R}^d$ endowed with the Euclidean metric. Set $\mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$ to be the set of first degree polynomials (affine functions) mapping $\mathbb{R}^d$ to $\mathbb{R}$. We take $Z = \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$, and write each $P \in \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$ in the following form:

$$P(a) = p_0 + D_0p \cdot a, \quad p_0 \in \mathbb{R},\ D_0p \in \mathbb{R}^d,\ a \in \mathbb{R}^d.$$

For any $P, Q \in \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$, we then define d_Z as:

$$d_Z(P,Q) \triangleq |p_0 - q_0| + \|D_0p - D_0q\|,$$

where $|\cdot|$ is just the absolute value, and $\|\cdot\|$ is the Euclidean distance on $\mathbb{R}^d$.

For a function $f \in \mathscr{F}(\mathbb{X}, Z)$, we use the following notation (note, as usual, $E \subset \mathbb{X}$):

$$\begin{aligned} f : E &\to \mathscr{P}^1(\mathbb{R}^d, \mathbb{R}) \\ x &\mapsto f(x)(a) = f_x + D_x f \cdot (a - x), \end{aligned}$$

where $f_x \in \mathbb{R}$, $D_x f \in \mathbb{R}^d$, and $a \in \mathbb{R}^d$ is the evaluation variable of the polynomial $f(x)$. Define the functional Φ as:

$$\begin{aligned} \Phi(f; x, y) = \Gamma^1(f; x, y) &\triangleq 2 \sup_{a \in \mathbb{R}^d} \frac{|f(x)(a) - f(y)(a)|}{\|x - a\|^2 + \|y - a\|^2}, \\ \Phi(f; E) = \Gamma^1(f; E) &\triangleq \sup_{\substack{x, y \in E \\ x \neq y}} \Gamma^1(f; x, y). \end{aligned}$$

Using the results contained in [20], one can show that for these two metric spaces and for this definition of Φ, that properties (P_0)-(P_5) are satisfied; the full details are given in Appendix A. In particular, there exists an extension $U : \mathbb{R}^d \to \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$, $U(x)(a) = U_x + D_x U \cdot (a - x)$, such that $U(x) = f(x)$ for all $x \in E$ and $\Phi(U; \mathbb{R}^d) = \Phi(f; E)$. Furthermore, define the function $F : \mathbb{R}^d \to \mathbb{R}$ as $F(x) = U_x$ for all $x \in \mathbb{R}^d$. Note that $F \in C^{1,1}(\mathbb{R}^d)$, and set for each $x \in \mathbb{R}^d$, $J_x F(a) \triangleq F(x) + \nabla F(x) \cdot (a - x) \in \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$ to be the first order Taylor expansion of F around x. Then F satisfies the following properties:

1. $J_x F = f(x)$ for all $x \in E$.
2. $\mathrm{Lip}(\nabla F) = \Gamma^1(f; E)$.
3. If $\widetilde{F} \in C^{1,1}(\mathbb{R}^d)$ also satisfies $J_x \widetilde{F} = f(x)$ for all $x \in E$, then $\mathrm{Lip}(\nabla F) \leq \mathrm{Lip}(\nabla \widetilde{F})$.

Thus F is the extension of the 1-field f with minimum Lipschitz derivative (see [20] for the proofs and a complete explanation). The 1-field U is the corresponding set of jets of F. For an explicit construction of F when E is finite we refer the reader to [23].

2.4.5 m-fields

A similar result for m-fields, where $m \geq 2$, is an open problem. In particular, it is unknown what the correct corresponding functional $\Phi = \Gamma^m$ is. It seems plausible, though, that such a functional will satisfy the properties (P_0)-(P_5).

2.5 Main theorem

The AMLE condition (6) is for any open set off of the domain of the initial function f. In our analysis, we look at subfamily of open sets that approximates the family of all open sets. In particular, we look at finite unions of open balls. The number of balls in a particular union is capped by a universal constant, and furthermore, the radius of each ball must also be larger than some constant. For any $\rho > 0$ and $N_0 \in \mathbb{N}$, define such a collection as:

$$\mathscr{O}(\rho, N_0) \triangleq \left\{ \Omega = \bigcup_{n=1}^{N} B(x_n; r_n) \mid x_n \in \mathbb{X}, \;\; r_n \geq \rho, \;\; N \leq N_0 \right\}.$$

Note that as $\rho \to 0$ and $N_0 \to \infty$, $\mathscr{O}(\rho,N_0)$ contains all open sets if $(\mathbb{X},d_{\mathbb{X}})$ is compact. We shall always use Ω to denote sets taken from $\mathscr{O}(\rho,N_0)$. For any such set, we use $\mathscr{R}(\Omega)$ to denote the collection of balls that make up Ω:

$$\mathscr{R}(\Omega) = \left\{ B(x_n;r_n) \mid n = 1,\ldots,N,\ \ \Omega = \bigcup_{n=1}^{N} B(x_n;r_n) \right\}.$$

We also define, for any $f \in \mathscr{F}(\mathbb{X},Z)$, any open $V \subset \operatorname{dom}(f)$, and any $\alpha > 0$, the following approximation of $\Phi(f;V)$:

$$\Psi(f;V;\alpha) \triangleq \sup\left\{\Phi(f;x,y) \mid B(x;d_{\mathbb{X}}(x,y)) \subset V,\ \ d_{\mathbb{X}}(x,\partial V) \geq \alpha\right\}.$$

Using these two approximations, our primary result is the following:

Theorem 1 *Let $(\mathbb{X},d_{\mathbb{X}})$ and (Z,d_Z) be metric spaces satisfying the assumptions of Section 2.1, let Φ be a functional satisfying properties (P_0)-(P_5), and let $X \in \mathscr{K}(\mathbb{X})$. Given $f \in \mathscr{F}_{\Phi}(X,Z)$, $\rho > 0$, $N_0 \in \mathbb{N}$, $\alpha > 0$, and $\sigma_0 > 0$, there exists $U = U(f,\rho,N_0,\alpha,\sigma_0) \in \mathscr{F}_{\Phi}(X,Z)$ such that*

1. *U is a minimal extension of f to X; that is,*

$$\begin{aligned}
&\operatorname{dom}(U) = X,\\
&U(x) = f(x), \textit{ for all } x \in \operatorname{dom}(f),\\
&\Phi(U;X) = \Phi(f;\operatorname{dom}(f)).
\end{aligned}$$

2. *The following quasi-AMLE condition is satisfied on X:*

$$\Psi(U;\Omega;\alpha) - \Phi(U;\partial\Omega) < \sigma_0, \quad \textit{for all } \Omega \in \mathscr{O}(\rho,N_0),\ \ \Omega \subset X \setminus \operatorname{dom}(f). \tag{7}$$

We call such extensions quasi-AMLEs, and view them as a first step toward proving the existence of AMLEs under these general conditions. We note that there are essentially four areas of approximation. The first is that we extend to an arbitrary, but fixed compact set $X \subset \mathbb{X}$ as opposed to the entire space. The second was already mentioned; rather than look at all open sets, we look at those belonging to $\mathscr{O}(\rho,N_0)$. Since X is compact, as $\rho \to 0$ and $N_0 \to \infty$, $\mathscr{O}(\rho,N_0)$ will contain all open sets in X. Third, we allow ourselves a certain amount of error with the parameter σ_0. As $\sigma_0 \to 0$, the values of the Lipschitz constants on Ω and $\partial\Omega$ should coincide. The last part of the approximation is the use of the functional Ψ to approximate Φ on each $\Omega \in \mathscr{O}(\rho,N_0)$. While this may at first not seem as natural as the other areas of approximation, the following proposition shows that in fact Ψ works rather well in the context of the AMLE problem.

Proposition 1 *Let $f \in \mathscr{F}_{\Phi}(\mathbb{X},Z)$. For any open $V \subset\subset \operatorname{dom}(f)$ and $\alpha \geq 0$, let us define*

$$V_\alpha \triangleq \{x \in V \mid d_{\mathbb{X}}(x,\partial V) \geq \alpha\}.$$

Then for all $\alpha > 0$ and for all open $V \subset\subset \operatorname{dom}(f)$,

$$\Phi(f;V_\alpha) \leq \max\{\Psi(f;V;\alpha), \Phi(u;\partial V_\alpha)\}, \tag{8}$$

and

$$\Phi(f;V) = \max\{\Psi(f;V;0), \Phi(f;\partial V)\}. \tag{9}$$

Proof See Appendix B.

Proposition 1, along with the discussion immediately preceding it, seems to indicate that if one were able to pass through the various limits to obtain $U(f,\rho,N_0,\alpha,\sigma_0) \to U(f)$ as $\rho \to 0$, $N_0 \to \infty$, $\alpha \to 0$, and $\sigma_0 \to 0$, then one would have a general theorem of existence of AMLEs for suitable pairs of metric spaces and Lipschitz-type functionals. Whether such a procedure is in fact possible, though, is yet to be determined.

The proof of Theorem 1 is given in Section 3, with the relevant lemmas stated and proved in Section 4. The main ideas of the proof are as follows. Using (P_2), we can find a minimal extension $U_0 \in \mathscr{F}_\Phi(X,Z)$ of f with $\mathrm{dom}(U_0) = X$. If such an extension also satisfies (7), then we take $U = U_0$ and we are finished. If, on the other hand, U_0 does not satisfy (7), then there must be some $\Omega_1 \in \mathscr{O}(\rho,N_0)$, $\Omega_1 \subset X \setminus \mathrm{dom}(f)$, for which $\Psi(U_0;\Omega_1;\alpha) - \Phi(U_0;\partial\Omega_1) \geq \sigma_0$. We derive a new minimal extension $U_1 \in \mathscr{F}_\Phi(X,Z)$ of f from U_0 by correcting U_0 on Ω_1. To perform the correction, we restrict ourselves to $U_0|_{\partial\Omega_1}$, and extend this function to Ω_1 using once again (P_2). We then patch this extension into U_0, giving us U_1. We then ask if U_1 satisfies (7). If it does, we take $U = U_1$ and we are finished. If it does not, we repeat the procedure just outlined. The main work of the proof then goes into showing that the repetition of such a procedure must end after a finite number of iterations.

It is also interesting to note that the extension procedure itself is a “black box.” We do not have any knowledge of the behavior of the extension outside of (P_0)-(P_5), only that it exists. We then refine this extension by using local extensions to correct in areas that do not satisfy the quasi-AMLE condition. The proof then is not about the extension of functions, but rather the refinement of such extensions.

3 Proof of Theorem 1: Existence of quasi-AMLE's

In this section we outline the key parts of the proof of Theorem 1. We begin by defining a local correction operator that we will use repeatedly.

3.1 Definition of the correction operator H

Definition 5 Let $X \in \mathscr{K}(\mathbb{X})$, $f \in \mathscr{F}_\Phi(X,Z)$, and $\Omega \in \mathscr{O}(\rho,N_0)$ with $\overline{\Omega} \subset \mathrm{dom}(f)$. By (P_2) there exists a $F \in \mathscr{F}_\Phi(X,Z)$ with $\mathrm{dom}(F) = \overline{\Omega}$ such that

$$F(x) = f(x), \text{ for all } x \in \partial\Omega, \\ \Phi(F;\Omega) = \Phi(f;\partial\Omega). \tag{10}$$

Given such an f and Ω, define the operator H as:

$$H(f;\Omega)(x) \triangleq F(x), \quad \text{for all } x \in \Omega. \tag{11}$$

3.2 A sequence of total, minimal extensions

Fix the metric spaces $(\mathbb{X}, d_{\mathbb{X}})$ and (Z, d_Z), the Lipschitz functional Φ, the compact domain $X \in \mathscr{K}(\mathbb{X})$, as well as $f \in \mathscr{F}_\Phi(X,Z)$, $\rho > 0$, $N_0 \in \mathbb{N}$, $\alpha > 0$, $\sigma_0 > 0$. Set:

$$K \triangleq \Phi(f; \mathrm{dom}(f)).$$

Using (P_2), let $U_0 \in \mathscr{F}_\Phi(X,Z)$ be a minimal extension of f to all of X; recall that this means:

$$\begin{aligned}
&\mathrm{dom}(U_0) = X,\\
&U_0(x) = f(x), \text{ for all } x \in \mathrm{dom}(f),\\
&\Phi(U_0; X) = \Phi(f; \mathrm{dom}(f)).
\end{aligned}$$

We are going to recursively construct a sequence $\{U_n\}_{n\in\mathbb{N}}$ of minimal extensions of f to X. First, for any $n \in \mathbb{N}$, define

$$\Delta_n \triangleq \{\Omega \in \mathscr{O}(\rho, N_0) \mid \Psi(U_n; \Omega; \alpha) - \Phi(U_n; \partial\Omega) \geq \sigma_0, \ \ \Omega \subset X \setminus \mathrm{dom}(f)\}.$$

The set Δ_n contains all admissible open sets for which the extension U_n violates the quasi-AMLE condition. If $\Delta_n = \emptyset$, then we can take $U = U_n$ and we are finished.

If, on the other hand, $\Delta_n \neq \emptyset$, then to obtain U_{n+1} we take U_n and pick any $\Omega_{n+1} \in \Delta_n$ and set

$$U_{n+1}(x) \triangleq \begin{cases} H(U_n; \Omega_{n+1})(x), & \text{if } x \in \Omega_{n+1},\\ U_n(x) & \text{if } x \in X \setminus \Omega_{n+1}, \end{cases}$$

where H was defined in Section 3.1. Thus, along with $\{U_n\}_{n\in\mathbb{N}}$, we also have a sequence of refining sets $\{\Omega_n\}_{n\in\mathbb{N}^*}$ such that $\Omega_n \in \mathscr{O}(\rho, N_0)$, $\Omega_n \subset X \setminus \mathrm{dom}(f)$, and $\Omega_n \in \Delta_{n-1}$ for all $n \in \mathbb{N}^*$.

Since $\mathrm{dom}(U_0) = X$, and since $\Omega_n \subset X \setminus \mathrm{dom}(f)$, we see by construction that $\mathrm{dom}(U_n) = X$ for all $n \in \mathbb{N}$. By the arguments in Section 4.1 and Lemma 3 contained within, we see that each of the functions U_n is also a minimal extension of f.

3.3 Reducing the Lipschitz constant on the refining sets $\{\Omega_n\}_{n\in\mathbb{N}^*}$

Since each of the functions U_n is a minimal extension of $f \in \mathscr{F}_\Phi(X,Z)$, we have

$$\Phi(U_n; X) = K, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Furthermore, since $\Omega_{n+1} \in \Delta_n$, we have by definition,

$$\Psi(U_n; \Omega_{n+1}; \alpha) - \Phi(U_n; \partial\Omega_{n+1}) \geq \sigma_0. \tag{13}$$

Using the definition of the operator H and (13), we also have for any $n \in \mathbb{N}^*$,

$$\Phi(U_n; \Omega_n) = \Phi(H(U_{n-1}; \Omega_n); \Omega_n) = \Phi(U_{n-1}; \partial\Omega_n) \leq \Psi(U_{n-1}; \Omega_n; \alpha) - \sigma_0. \tag{14}$$

Furthermore, combining (12) and (14), and using property (P_1) as well as the definition of Ψ, one can arrive at the following:

$$\Phi(U_n; \Omega_n) \leq K - \sigma_0, \quad \text{for all } n \in \mathbb{N}^*. \tag{15}$$

Thus we see that locally on Ω_n, the total, minimal extension U_n is guaranteed to have Lipschitz constant bounded by $K-\sigma_0$. In fact we can say much more.

Lemma 1 *The following property holds true for all $p \in \mathbb{N}^*$:*

$$\exists M_p \in \mathbb{N}^* : \forall n > M_p, \quad \Phi(U_n;\Omega_n) < K - p\frac{\sigma_0}{2}. \qquad (\mathbf{Q}_p)$$

The property $(\mathbf{Q}_p)$ is enough to prove Theorem 1. Indeed, if $\Delta_n \neq \emptyset$ for all $n \in \mathbb{N}$, then by $(\mathbf{Q}_p)$ one will have $\Phi(U_n;\Omega_n) < 0$ for n sufficiently large. However, by the definition of Φ one must have $\Phi(U_n;\Omega_n) \geq 0$, and so we have arrived at a contradiction. Now for the proof of Lemma 1.

Proof We prove $(\mathbf{Q}_p)$ by induction. By (15), it is clearly true for $p=1$. Let $p \geq 2$ and suppose that $(\mathbf{Q}_{p-1})$ is true; we wish to show that $(\mathbf{Q}_p)$ is true as well. Let M_{p-1} be an integer satisfying $(\mathbf{Q}_{p-1})$ and assume that $\Delta_{M_{p-1}} \neq \emptyset$. Let us define the following sets:

$$\begin{aligned}
\mathscr{A}_{p,n} &\triangleq \bigcup \{\Omega_m \mid M_{p-1} < m \leq n\}, \\
\mathscr{A}_{p,\infty} &\triangleq \bigcup \{\Omega_m \mid M_{p-1} < m\}, \\
\widetilde{\mathscr{R}}(\mathscr{A}_{p,\infty}) &\triangleq \{B(x;r) \mid \exists m > M_{p-1} \text{ with } B(x;r) \in \mathscr{R}(\Omega_m)\}.
\end{aligned}$$

The closure of each set $\mathscr{A}_{p,n}$ is compact and the sequence $\{\mathscr{A}_{p,n}\}_{n>M_{p-1}}$ is monotonic under inclusion and converges to $\mathscr{A}_{p,\infty}$ in Hausdorff distance as $n \to \infty$. In particular, for $\varepsilon > 0$, there exists $N_p > M_{p-1}$ such that

$$\delta(\mathscr{A}_{p,N_p}, \mathscr{A}_{p,\infty}) \leq \varepsilon,$$

where δ is the Hausdorff distance.

Now apply the Geometrical Lemma 6 to the sets $\mathscr{A}_{p,n}$ and $\mathscr{A}_{p,\infty}$ with $\beta = \alpha - \varepsilon$. One obtains $N_\varepsilon \in \mathbb{N}$ such that

$$\forall B(x;r), \text{ if } r \geq \alpha - \varepsilon \text{ and } B(x;r) \subset \mathscr{A}_{p,\infty}, \text{ then } B(x;r-\varepsilon) \subset \mathscr{A}_{p,N_\varepsilon}. \qquad (16)$$

Take $M_p \triangleq \max\{N_p, N_\varepsilon\}$. One can then obtain the following lemma, which is essentially a corollary of (16).

Lemma 2 *For all $n > M_p$ and for all $B(x;r) \subset \Omega_n$ with $d_{\mathbb{X}}(x,\partial\Omega_n) \geq \alpha$ and $r < \alpha$, we have*

$$\textit{if } B(x;r) \not\subset \mathscr{A}_{p,M_p}, \textit{ then } r \geq \alpha - \varepsilon.$$

Proof Let $B(x;r)$ be a ball that satisfies the hypothesis of the lemma and suppose $B(x;r) \not\subset \mathscr{A}_{p,M_p}$. Since $d_{\mathbb{X}}(x,\partial\Omega_n) \geq \alpha$ and $B(x;r) \subset B(x;\alpha) \subset \Omega_n$, we have $B(x;\alpha) \not\subset \mathscr{A}_{p,M_p}$. On the other hand, $B(x;\alpha) \subset \mathscr{A}_{p,\infty}$ and (trivially) $\alpha \geq \alpha - \varepsilon$, so by (16), $B(x;\alpha-\varepsilon) \subset \mathscr{A}_{p,M_p}$. Therefore $r \geq \alpha - \varepsilon$. □

Now let us use the inductive hypothesis $(\mathbf{Q}_{p-1})$. Let $n > M_p$ and let $x, y \in \Omega_n$ such that $B(x; d_{\mathbb{X}}(x,y)) \subset \Omega_n$ with $d_{\mathbb{X}}(x,\partial\Omega_n) \geq \alpha$.

Case 1 Suppose that $B(x;d_{\mathbb{X}}(x,y)) \subset \mathscr{A}_{p,M_p}$. In this case we apply the Customs Lemma 5 with $\mathscr{A} = \mathscr{A}_{p,n-1}$. Since $n > M_p > M_{p-1}$, we are assured by the inductive hypothesis that

$$\Phi(U_j;\Omega_j) < K - (p-1)\frac{\sigma_0}{2}, \quad \text{for all } j = M_{p-1}+1,\ldots,n-1.$$

Thus we can conclude from the Customs Lemma that

$$\Phi(U_{n-1};x,y) \leq K - (p-1)\frac{\sigma_0}{2}. \tag{17}$$

That completes the first case.

Case 2 Suppose that $B(x;d_{\mathbb{X}}(x,y)) \not\subset \mathscr{A}_{p,M_p}$. By Lemma 2, we know that $d_{\mathbb{X}}(x,y) \geq \alpha - \varepsilon$. Thus, by (16), we have $B(x;d_{\mathbb{X}}(x,y)-2\varepsilon) \subset \mathscr{A}_{p,M_p}$.

Let $\gamma \in \Gamma(x,y)$ be the curve satisfying (P_3) and set

$$y_1 \triangleq \partial B(x;d_{\mathbb{X}}(x,y)-2\varepsilon) \cap \gamma.$$

Write $\Phi(U_{n-1};x,y)$ in the form:

$$\Phi(U_{n-1};x,y) = \Phi(U_{n-1};x,y) - \Phi(U_{n-1};x,y_1) + \Phi(U_{n-1};x,y_1).$$

Using (P_4) and the fact that $d_{\mathbb{X}}(x,y) \geq \alpha - \varepsilon$, there exists a constant $C(\varepsilon,\alpha)$ satisfying $C(\varepsilon,\alpha) \to 0$ as $\varepsilon \to 0$ such that

$$\Phi(U_{n-1};x,y) - \Phi(U_{n-1};x,y_1) \leq C(\varepsilon,\alpha). \tag{18}$$

Moreover, since $B(x;d_{\mathbb{X}}(x,y_1)) \subset \mathscr{A}_{p,M_p}$, we can apply the Customs Lemma 5 along with the inductive hypothesis $(\mathbf{Q}_{p-1})$ (as in the first case) to conclude that

$$\Phi(U_{n-1};x,y_1) \leq K - (p-1)\frac{\sigma_0}{2}. \tag{19}$$

Combining (18) and (19) we obtain

$$\Phi(U_{n-1};x,y) \leq K - (p-1)\frac{\sigma_0}{2} + C(\varepsilon,\alpha).$$

Since we can choose ε such that $C(\varepsilon,\alpha) \leq \sigma_0/2$, we have

$$\Phi(U_{n-1};x,y) \leq K - p\frac{\sigma_0}{2} + \sigma_0. \tag{20}$$

That completes the second case.

Now using (17) in the first case and (20) in the second case we obtain

$$\Psi(U_{n-1};\Omega_n;\alpha) \leq K - p\frac{\sigma_0}{2} + \sigma_0. \tag{21}$$

Combining (14) with (21) we can complete the proof:

$$\Phi(U_n;\Omega_n) \leq \Psi(U_{n-1};\Omega_n;\alpha) - \sigma_0 \leq K - p\frac{\sigma_0}{2}.$$

□

4 Lemmas used in the proof Theorem 1

4.1 The operator H preserves the Lipschitz constant

In this section we prove that the sequence of extensions $\{U_n\}_{n\in\mathbb{N}}$ constructed in Section 3.2 are all minimal extensions of the original function $f \in \mathscr{F}_{\Phi}(X,Z)$. Recall that by construction, U_0 is a minimal extension of f, and each U_n is an extension of f, so it remains to show that each U_n, for $n \in \mathbb{N}^*$, is minimal. In particular, if we show that the construction preserves or lowers the Lipschitz constant of the extension from U_n to U_{n+1} then we are finished. The following lemma does just that.

Lemma 3 *Let $F_0 \in \mathscr{F}_{\Phi}(X,Z)$ with* $\mathrm{dom}(F_0) = X$ *and let $\Omega \in \mathscr{O}(\rho, N_0)$. Define $F_1 \in \mathscr{F}_{\Phi}(X,Z)$ as:*

$$F_1(x) \triangleq \begin{cases} H(F_0;\Omega)(x), & \text{if } x \in \Omega, \\ F_0(x), & \text{if } x \in X \setminus \Omega. \end{cases}$$

Then,

$$\Phi(F_1;X) \le \Phi(F_0;X).$$

Proof We utilize properties (P_1) and (P_3). By (P_1), it is enough to consider the evaluation of $\Phi(F_1;x,y)$ for an arbitrary pair of points $x,y \in X$. We have three cases:

Case 1 If $x,y \in X \setminus \Omega$, then by the definition of F_1 and (P_1) (applied to F_0) we have:

$$\Phi(F_1;x,y) = \Phi(F_0;x,y) \le \Phi(F_0;X).$$

Case 2 If $x,y \in \Omega$, then by the definition of F_1, the definition of H, and property (P_1), we have:

$$\Phi(F_1;x,y) = \Phi(H(F_0;\Omega);x,y) \le \Phi(F_0;\partial\Omega) \le \Phi(F_0;X).$$

Case 3 Suppose that $x \in X \setminus \Omega$ and $y \in \Omega$. Assume, for now, that $B_{1/2}(x,y) \subset X$. By (P_3) there exists a curve $\gamma \in \Gamma(x,y)$ such that

$$\Phi(F_1;x,y) \le \inf_{t\in[0,1]} \max\{\Phi(F_1;x,\gamma(t)), \Phi(F_1;\gamma(t),y)\}.$$

Let $t_0 \in [0,1]$ be such that $\gamma(t_0) \in \partial\Omega$. Then, utilizing (P_3), the definition of F_1, the definition of H, and (P_1), one has:

$$\begin{aligned} \Phi(F_1;x,y) &\le \max\{\Phi(F_1;x,\gamma(t_0)), \Phi(F_1;\gamma(t_0),y)\} \\ &= \max\{\Phi(F_0;x,\gamma(t_0)), \Phi(H(F_0;\Omega);\gamma(t_0),y)\} \\ &\le \Phi(F_0;X). \end{aligned}$$

If $B_{1/2}(x,y) \not\subseteq X$, then we can replace X by a larger compact set $\widetilde{X} \subset \mathbb{X}$ that does contain $B_{1/2}(x,y)$. By (P_2), extend F_0 to a function $\widetilde{F_0}$ with $\mathrm{dom}(\widetilde{F_0}) = \widetilde{X}$ such that

$$\begin{aligned} &\widetilde{F_0}(x) = F_0(x), \quad \text{for all } x \in X, \\ &\Phi(\widetilde{F_0};\widetilde{X}) = \Phi(F_0;X). \end{aligned}$$

Define $\widetilde{F}_1$ analogously to F_1:

$$\widetilde{F}_1(x) \triangleq \begin{cases} H(\widetilde{F}_0;\Omega)(x), & \text{if } x \in \Omega, \\ \widetilde{F}_0(x), & \text{if } x \in \widetilde{X} \setminus \Omega. \end{cases}$$

Note that $\widetilde{F}_1|_X \equiv F_1$, and furthermore, the analysis just completed at the beginning of case three applies to $\widetilde{F}_0$, $\widetilde{F}_1$, and $\widetilde{X}$ since $B_{1/2}(x,y) \subset \widetilde{X}$. Therefore,

$$\Phi(F_1;x,y) = \Phi(\widetilde{F}_1;x,y) \le \Phi(\widetilde{F}_0;\widetilde{X}) = \Phi(F_0;X).$$

□

4.2 Customs Lemma

In this section we prove the Customs Lemma, which is vital to the proof of the property ($\mathbf{Q}_p$) from Lemma 1. Throughout this section we shall make use of the construction of the sequence of extensions $\{U_n\}_{n\in\mathbb{N}}$, which we repeat here.

Let $U_0 \in \mathscr{F}_\Phi(X,Z)$ with $\operatorname{dom}(U_0) = X$ and $n \in \mathbb{N}^*$. Set

$$\Lambda \triangleq \{\Omega_j \mid \Omega_j \in \mathscr{O}(\rho, N_0), \;\; j = 1,\ldots,n\},$$

and define:

$$\mathscr{A} \triangleq \bigcup_{j=1}^{n} \Omega_j.$$

Let $\{U_j\}_{j=1}^n \subset \mathscr{F}_\Phi(X,Z)$ be a collection of functions defined as:

$$U_{j+1}(x) \triangleq \begin{cases} H(U_j;\Omega_{j+1})(x), & \text{if } x \in \Omega_{j+1}, \\ U_j(x), & \text{if } x \in X \setminus \Omega_{j+1}, \end{cases} \qquad \text{for all } j = 0,\ldots,n-1.$$

We shall need the following lemma first.

Lemma 4 *Let $x \in \mathscr{A}$. Then there exists $\sigma > 0$ so that $B(x;\sigma) \subset \mathscr{A}$, and for each $b \in B(x;\sigma)$, there exists $j \in \{1,\ldots,n\}$ such that $x, b \in \Omega_j$, $U_n(x) = U_j(x)$, and $U_n(b) = U_j(b)$.*

Proof To begin, set

$$\eta_1 \triangleq \sup\{r > 0 \mid B(x;r) \subset \mathscr{A}\},$$

noting that $\mathscr{A}$ is open and so $\eta_1 > 0$. Define the following two sets of indices:

$$I^+ \triangleq \{j \in \{1,\ldots,n\} \mid x \in \Omega_j\},$$
$$I^- \triangleq \{j \in \{1,\ldots,n\} \mid x \notin \Omega_j\}.$$

The set I^+ is nonempty since $x \in \mathscr{A}$. So we can additionally define

$$j^+ \triangleq \max_{j \in I^+} j.$$

On the other hand, I^- may be empty. If it is not, then we define $\ell_j \triangleq d_{\mathbb{X}}(x,\Omega_j)$ for each $j \in I^-$, and set

$$\eta_2 \triangleq \frac{1}{2}\min\{\ell_j \mid j \in I^-\}.$$

Finally, we take η to be:

$$\eta \triangleq \begin{cases} \min\{\eta_1,\eta_2\}, & \text{if } I^- \neq \emptyset, \\ \eta_1, & \text{if } I^- = \emptyset. \end{cases}$$

Note that $\eta > 0$; we also have:

$$B(x;\eta) \cap \bigcup_{j\in I^-} \Omega_j = \emptyset \qquad \text{and} \qquad B(x;\eta) \subset \bigcup_{j\in I^+} \Omega_j. \tag{22}$$

Now let

$$J \triangleq \{j \in I^+ \mid U_n(x) = U_j(x)\}.$$

Clearly $j^+ \in J$, and so this set is nonempty. We use it to define the following:

$$\Sigma \triangleq \{b \in B(x;\eta) \mid \exists\, j \in J : U_n(b) = U_j(b),\ \ b \in \Omega_j\}.$$

The set Σ is nonempty since $B(x;\eta) \cap \Omega_{j^+} \subset \Sigma$.

To prove the lemma, it is enough to show that $x \in \mathring{\Sigma}$. Indeed, if $x \in \mathring{\Sigma}$, then there exists a $\sigma > 0$ such that $B(x;\sigma) \subset \mathring{\Sigma}$. Then for each $b \in B(x;\sigma)$, there exists $j \in J$ such that $U_n(b) = U_j(b)$ (by the definition of Σ) and $U_n(x) = U_j(x)$ (by the definition of J).

We prove that $x \in \mathring{\Sigma}$ by contradiction. Suppose that $x \notin \mathring{\Sigma}$. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence which converges to x that satisfies the following property:

$$\forall k \in \mathbb{N},\ \ z_k \notin \Sigma \text{ and } z_k \in B(x;\eta).$$

Define:

$$I_k^+ \triangleq \{j \in I^+ \mid z_k \in \Omega_j\}, \quad \text{for all } k \in \mathbb{N}.$$

By the remark given in (22) we see that I_k^+ is nonempty for each $k \in \mathbb{N}$. Thus we can define

$$j_k \triangleq \max_{j\in I_k} j.$$

Since $I^+ \setminus J$ has a finite number of elements, there exists $i_0 \in I^+ \setminus J$ and a subsequence $\{z_{\phi(k)}\}_{k\in\mathbb{N}} \subset \{z_k\}_{k\in\mathbb{N}}$ that converges to x such that

$$\forall k \in \mathbb{N},\ \ j_{\phi(k)} = i_0.$$

By the definition of I_k^+ and using the fact that i_0 is the largest element of $I_{\phi(k)}^+$ for each $k \in \mathbb{N}$, we have

$$\forall k \in \mathbb{N},\ \ U_n(z_{\phi(k)}) = U_{i_0}(z_{\phi(k)}) \text{ and } z_{\phi(k)} \in \Omega_{i_0}.$$

Since the functions U_j are continuous by (P_5), we have

$$\lim_{k\to\infty} U_n(z_{\phi(k)}) = U_n(x),$$

and

$$\lim_{k\to\infty} U_{i_0}(z_{\phi(k)}) = U_{i_0}(x).$$

Thus

$$U_n(x) = U_{i_0}(x).$$

But then $i_0 \in J$, which in turn implies that $z_{\phi(k)} \in \Sigma$ for all $k \in \mathbb{N}$. Thus we have a contradiction, and so $x \in \mathring{\Sigma}$. □

Lemma 5 (Customs Lemma) *If there exists some constant $C \geq 0$ such that*

$$\Phi(U_j;\Omega_j) \leq C, \quad \textit{for all } j = 1,\dots,n,$$

then for all $x, y \in \mathscr{A}$ with $B(x; d_{\mathbb{X}}(x,y)) \subset \mathscr{A}$,

$$\Phi(U_n;x,y) \leq C.$$

Proof Let $x \in \mathscr{A}$ and define

$$\mathscr{A}(x) \triangleq \{y \in \mathscr{A} \mid B(x;d_{\mathbb{X}}(x,y)) \subset \mathscr{A}\}.$$

The set $\mathscr{A}(x)$ is a ball centered at x. Furthermore, using Lemma 4, there exists a $\sigma > 0$ and a corresponding ball $B(x;\sigma) \subset \mathscr{A}$ such that

$$\Phi(U_n;x,b) \leq C, \quad \text{for all } b \in B(x;\sigma).$$

In particular, we have

$$\Phi(U_n;x,y) \leq C, \quad \text{for all } y \in B(x;\sigma) \cap \mathscr{A}(x). \tag{23}$$

Consider the set

$$\mathscr{A}_\sigma(x) \triangleq \mathscr{A}(x) \setminus (B(x;\sigma) \cap \mathscr{A}(x)).$$

The set $\mathscr{A}_\sigma(x)$ contains those points $y \in \mathscr{A}(x)$ for which we do not yet have an upper bound for $\Phi(U_n;x,y)$. Let

$$M \triangleq \sup_{y\in\mathscr{A}_\sigma(x)} \Phi(U_n;x,y).$$

If we can show that $M \leq C$, then we are finished since we took x to be an arbitrary point of $\mathscr{A}$. By (P_4), the function $y \in \mathscr{A}_\sigma(x) \mapsto \Phi(U_n;x,y)$ is continuous. Thus,

$$M = \sup_{y\in\overline{\mathscr{A}_\sigma(x)}} \Phi(U_n;x,y).$$

Since X is compact, $\overline{\mathscr{A}_\sigma(x)}$ is compact as well, and so the set

$$\mathscr{S} \triangleq \{y \in \overline{\mathscr{A}_\sigma(x)} \mid \Phi(U_n;x,y) = M\}$$

is nonempty. We select $y_0 \in \mathscr{S}$ such that

$$d_{\mathbb{X}}(x,y_0) \leq d_{\mathbb{X}}(x,y), \quad \text{for all } y \in \mathscr{S}. \tag{24}$$

Since $\mathscr{S}$ is closed and a subset of $\overline{\mathscr{A}_\sigma(x)}$, it is also compact. Furthermore, the function $y \in \mathscr{S} \mapsto d_{\mathbb{X}}(x,y)$ is continuous, and so the point y_0 must exist. It is, by definition, the point in $\overline{\mathscr{A}_\sigma(x)}$ that not only achieves the maximum value of the function $y \in \overline{\mathscr{A}_\sigma(x)} \mapsto \Phi(U_n;x,y)$, but also, amongst all such points, it is the one closest to x. Thus we have reduced the problem to showing that $M = \Phi(U_n;x,y_0) \leq C$.

We claim that it is sufficient to show the following: there exists a point $y_1 \in \mathscr{A}(x)$ such that $d_{\mathbb{X}}(x,y_1) < d_{\mathbb{X}}(x,y_0)$, and furthermore satisfies:

$$M = \Phi(U_n;x,y_0) \leq \max\{\Phi(U_n;x,y_1), C\}. \tag{25}$$

Indeed, if such a point were to exist, then we could complete the proof in the following way. If C is the max of the right hand side of (25), then clearly we are finished. If, on the other hand, $\Phi(U_n;x,y_1)$ is the max, then we have two cases to consider. If $d_{\mathbb{X}}(x,y_1) < \sigma$, then $y_1 \in B(x;\sigma) \cap \mathscr{A}(x)$, and so by (23) we know that $\Phi(U_n;x,y_1) \leq C$. Alternatively, if $d_{\mathbb{X}}(x,y_1) \geq \sigma$, then $y_1 \in \mathscr{A}_\sigma(x)$ and by the definition of M we have $\Phi(U_n;x,y_1) \leq M$, which by (25) implies that $\Phi(U_n;x,y_1) = M$. But y_0 is the closest point to x for which the function $y \in \overline{\mathscr{A}_\sigma(x)} \mapsto \Phi(U_n;x,y)$ achieves the maximum M. Thus we have arrived at a contradiction.

Now we are left with the task of showing the existence of such a point y_1. Apply Lemma 4 to the point y_0 to obtain a radius σ' such that $B(y_0;\sigma') \subset \mathscr{A}$ and for each $b \in B(y_0;\sigma')$, one has $\Phi(U_n;y_0,b) \leq C$. Since $y_0 \in \mathscr{A}_\sigma(x) \subset \mathscr{A}(x)$, we also know that $B(x;d_{\mathbb{X}}(x,y_0)) \subset \mathscr{A}(x) \subset \mathscr{A}$. Therefore $B_{1/2}(x,y_0) \subset B(x;d_{\mathbb{X}}(x,y_0)) \subset \mathscr{A}(x)$. Let $\gamma : [0,1] \to B_{1/2}(x,y_0)$ be the curve guaranteed to exist by (P_3), and take y_1 be the intersection point of γ with $\partial B(y_0;\sigma')$. Clearly $y_1 \in B_{1/2}(x,y_0) \subset \mathscr{A}(x)$, and furthermore it satisfies:

$$\begin{aligned}
\Phi(U_n;x,y_0) &\leq \inf_{t\in[0,1]} \max\{\Phi(U_n;x,\gamma(t)), \Phi(U_n;\gamma(t),y_0)\} \\
&\leq \max\{\Phi(U_n;x,y_1), \Phi(U_n;y_1,y_0)\} \\
&\leq \max\{\Phi(U_n;x,y_1), C\}.
\end{aligned}$$

Finally, using the monotonicity property of the curve γ, we see that $d_{\mathbb{X}}(x,y_1) < d_{\mathbb{X}}(x,y_0)$. □

4.3 Geometrical Lemma

Lemma 6 *Fix $\rho > 0$ and $\beta > 0$ with $\beta < \rho$. Let $\{B(x_n;r_n)\}_{n\in\mathbb{N}}$ be a set of balls contained in X. Suppose that $\forall n \in \mathbb{N}$, $r_n > \rho$. For $N \in \mathbb{N}$, let us define*

$$\mathscr{A}_N \triangleq \bigcup_{n\leq N} B(x_n;r_n) \quad \textit{and} \quad \mathscr{A}_\infty \triangleq \bigcup_{n\in\mathbb{N}} B(x_n;r_n).$$

Then $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall B(x;r)$, with $r \geq \beta$ and $B(x;r) \subset \mathscr{A}_\infty$, we have

$$B(x;r-\varepsilon) \subset \mathscr{A}_{N_\varepsilon}.$$

Proof Let $\varepsilon > 0$. Let us define for all $N \in \mathbb{N}$,

$$I_N \triangleq \{a \mid B(a;\beta - 2\varepsilon) \subset \mathscr{A}_N\} \quad \text{and} \quad I_\infty \triangleq \{b \mid B(b;\beta - 2\varepsilon) \subset \mathscr{A}_\infty\}.$$

We remark that $r_n > \rho > \beta - 2\varepsilon$ implies that

$$\mathscr{A}_N = \bigcup_{a \in I_N} B(a;\beta - 2\varepsilon) \quad \text{and} \quad \mathscr{A}_\infty = \bigcup_{b \in I_\infty} B(b;\beta - 2\varepsilon).$$

Let us define

$$\mathscr{A}_N^{\varepsilon/2} \triangleq \bigcup_{a \in I_N} B(a;\frac{\varepsilon}{2}) \quad \text{and} \quad \mathscr{A}_\infty^{\varepsilon/2} \triangleq \bigcup_{b \in I_\infty} B(b;\frac{\varepsilon}{2}).$$

The sequence $\{\mathscr{A}_N^{\varepsilon/2}\}_{N\in\mathbb{N}}$ is monotonic under inclusion and converges to $\mathscr{A}_\infty^{\varepsilon/2}$ in Hausdorff distance as $n \to \infty$. For $\varepsilon/2 > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\delta(\mathscr{A}_{N_\varepsilon}^{\varepsilon/2}, \mathscr{A}_\infty^{\varepsilon/2}) \leq \frac{\varepsilon}{2}. \tag{26}$$

Choose any ball $B(x;r) \subset \mathscr{A}_\infty$ with $r \geq \beta$ and define

$$J(x) \triangleq \{c \mid B(c;\beta - 3\varepsilon) \subset B(x;r-\varepsilon)\}.$$

We note, similar to earlier, that $r > \beta - 2\varepsilon$ implies we have $B(x;r-\varepsilon) = \bigcup_{c\in J(x)} B(c;\beta - 3\varepsilon)$. We will show that $B(x;r-\varepsilon) \subset \mathscr{A}_{N_\varepsilon}$.

Let $y \in B(x;r-\varepsilon)$ and choose $c \in J(x)$ such that $y \in B(c;\beta - 3\varepsilon)$. Since $B(c;\beta - 3\varepsilon) \subset B(x;r-\varepsilon)$ and $B(x;r) \subset \mathscr{A}_\infty$ we have

$$B(c;\beta - 2\varepsilon) \subset B(x;r) \subset \mathscr{A}_\infty.$$

Thus $c \in I_\infty$ and $c \in \mathscr{A}_\infty^{\varepsilon/2}$. Since $c \in \mathscr{A}_\infty^{\varepsilon/2}$, using (26), choose $z \in \mathscr{A}_{N_\varepsilon}^{\varepsilon/2}$ which satisfies

$$d_{\mathbb{X}}(c,z) \leq \frac{\varepsilon}{2}. \tag{27}$$

Moreover since $z \in \mathscr{A}_{N_\varepsilon}^{\varepsilon/2}$, choose $a \in I_{N_\varepsilon}$ which satisfies $z \in B(a;\varepsilon/2)$. We have

$$d_{\mathbb{X}}(z,a) \leq \frac{\varepsilon}{2}. \tag{28}$$

By (27) and (28),

$$d_{\mathbb{X}}(c,a) \leq d_{\mathbb{X}}(c,z) + d_{\mathbb{X}}(z,a) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \tag{29}$$

Since $y \in B(c;\beta - 3\varepsilon)$ we obtain

$$d_{\mathbb{X}}(y,a) \leq d_{\mathbb{X}}(y,c) + d_{\mathbb{X}}(c,a) \leq \beta - 3\varepsilon + \varepsilon \leq \beta - 2\varepsilon. \tag{30}$$

Since $a \in I_{N_\varepsilon}$ we conclude that $y \in \mathscr{A}_{N_\varepsilon}$. Therefore $B(x;r-\varepsilon) \subset \mathscr{A}_{N_\varepsilon}$ and the result is proved. □

5 Open questions and future directions

From here, there are several possible directions. The first was already mentioned earlier, and involves the behavior of the quasi-AMLE $U(f,\rho,N_0,\alpha,\sigma_0)$ as $\rho \to 0$, $N_0 \to \infty$, $\alpha \to 0$, and $\sigma_0 \to 0$. For the limits in α and σ_0 in particular, it seems that either more understanding or further exploitation of the geometrical relationship between $(\mathbb{X},d_{\mathbb{X}})$ and (Z,d_Z) is necessary. Should something of this nature be resolved, though, it would prove the existence of an AMLE under this general setup.

One may also wish to relax the assumptions on $(\mathbb{X},d_{\mathbb{X}})$. In the cases in which (4) is not equivalent to (3), distinctly new ideas are most likely necessary. In other cases of simpler relaxations, it may be possible to amend the arguments more easily.

A final possible question concerns the property (P_2). This property requires that an isometric extension exist for each $f \in \mathscr{F}_\Phi(\mathbb{X},Z)$; that is, that the Lipschitz constant is preserved perfectly. What if, however, one had the weaker condition that the Lipschitz constant be preserved up to some constant? In other words, suppose that we replace (P_2) with the following weaker condition:

(P_2') Isomorphic Lipschitz extension:
For all $f \in \mathscr{F}_\Phi(X,Z)$ and for all $D \subset X$ such that $\mathrm{dom}(f) \subset D$, there exists an extension $F : D \to Z$ such that

$$\Phi(F;D) \le C \cdot \Phi(f;\mathrm{dom}(f)), \tag{31}$$

where C depends on $(\mathbb{X},d_{\mathbb{X}})$ and (Z,d_Z).

Suppose then we wish to find an F satisfying (31) that also satisfies the AMLE condition to within a constant factor? The methods here, in which we correct locally, would be hard to adapt given that with each correction, we would lose a factor of C in (31).

A Proof that (P_0)-(P_5) hold for 1-fields

In this appendix we consider the case of 1-fields and the functional $\Phi = \Gamma^1$ first defined in Section 2.4.4. Recall that $(\mathbb{X},d_{\mathbb{X}}) = \mathbb{R}^d$ with $d_{\mathbb{X}}(x,y) = \|x-y\|$, where $\|\cdot\|$ is the Euclidean distance. The range (Z,d_Z) is taken to be $\mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$, with elements $P \in \mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$ given by $P(a) = p_0 + D_0 p \cdot a$, with $p_0 \in \mathbb{R}$, $D_0 p \in \mathbb{R}^d$, and $a \in \mathbb{R}^d$. The distance d_Z is defined as: $d_Z(P,Q) \triangleq |p_0 - q_0| + \|D_0 p - D_0 q\|$. For a function $f \in \mathscr{F}(\mathbb{X},Z)$, we use the notation $x \in \mathrm{dom}(f) \mapsto f(x)(a) = f_x + D_x f \cdot (x-a)$, where $f_x \in \mathbb{R}$, $D_x f \in \mathbb{R}^d$, and once again $a \in \mathbb{R}^d$. Note that $f(x) \in \mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$. The functional Φ is defined as:

$$\Phi(f;x,y) = \Gamma^1(f;x,y) \triangleq 2 \sup_{a \in \mathbb{R}^d} \frac{|f(x)(a) - f(y)(a)|}{\|x-a\|^2 + \|y-a\|^2}. \tag{32}$$

Rather than Φ, we shall write Γ^1 throughout the appendix. The goal is to show that the properties (P_0)-(P_5) hold for Γ^1 and the metric spaces $(\mathbb{X},d_{\mathbb{X}})$ and (Z,d_Z).

A.1 (P_0) and (P_1) for Γ^1

The property (P_0) (symmetry and nonnegative) is clear from the definition of Γ^1 in (32). The property (P_1) (pointwise evaluation) is by definition.

A.2 (P_2) for Γ^1

The property (P_2) (existence of a minimal extension to $\mathbb{X}$ for each $f \in \mathscr{F}_{\Gamma^1}(\mathbb{X},Z)$) is the main result of [20]. We refer the reader to that paper for the details.

A.3 (P_3) for Γ^1

Showing property (P_3), Chasles' inequality, requires a detailed study of the domain of uniqueness for a biponctual 1−field (i.e., when $\mathrm{dom}(f)$ consists of two points). Let $\mathscr{P}^m(\mathbb{R}^d,\mathbb{R})$ denote the space of polynomials of degree m mapping $\mathbb{R}^d$ to $\mathbb{R}$.

For $f \in \mathscr{F}_{\Gamma^1}(\mathbb{X},Z)$ and $x,y \in \mathrm{dom}(f)$, $x \neq y$ we define

$$A(f;x,y) \triangleq \frac{2(f_x - f_y) + (D_x f + D_y f)\cdot(y-x)}{\|x-y\|^2}$$

and

$$B(f;x,y) \triangleq \frac{\|D_x u - D_y u\|}{\|x-y\|}. \tag{33}$$

Using [20], Proposition 2.2, we have for any $D \subset \mathrm{dom}(f)$,

$$\Gamma^1(f;D) = \sup_{\substack{x,y\in D\\ x\neq y}} \left(\sqrt{A(f;x,y)^2 + B(f;x,y)^2} + |A(f;x,y)| \right). \tag{34}$$

For the remainder of this section, fix $f \in \mathscr{F}_{\Gamma^1}(\mathbb{X},Z)$, with $\mathrm{dom}(f) = \{x,y\}$, $x \neq y$, $f(x) = P_x$, $f(y) = P_y$, and set

$$M \triangleq \Gamma^1(f;\mathrm{dom}(f),\mathrm{dom}(f)). \tag{35}$$

Also, for an arbitrary pair of points $a,b \in \mathbb{R}^d$, let $[a,b]$ denote the closed line segment with end points a and b.

Proposition 2 *Let F be an extension of f such that $B_{1/2}(x,y) \subset \mathrm{dom}(F)$. Then there exists a point $c \in B_{1/2}(x,y)$ that depends only on f such that*

$$\Gamma^1(F;x,y) \leq \max\{\Gamma^1(F;x,a), \Gamma^1(F;a,y)\}, \textit{ for all } a \in [x,c] \cup [c,y]. \tag{36}$$

Remark 1 Proposition 2 implies that the operator Γ^1 satisfies the Chasles' inequality (property (P_3)). In particular, consider an arbitrary 1-field $g \in \mathscr{F}_{\Gamma^1}(\mathbb{X},Z)$ with $x,y \in \mathrm{dom}(g)$ such that $B_{1/2}(x,y) \subset \mathrm{dom}(g)$. Then g is trivially an extension of the 1-field $g|_{\{x,y\}}$, and so in particular satisfies (36). But this is the Chasles' inequality with $\gamma = [x,c] \cup [c,y]$.

To prove proposition 2 we will use the following lemma.

Lemma 7 *There exists $c \in B_{1/2}(x,y)$ and $s \in \{-1,1\}$ such that*

$$M = 2s\frac{P_x(c) - P_y(c)}{\|x-c\|^2 + \|y-c\|^2}.$$

Furthermore,

$$\begin{aligned} c &= \frac{x+y}{2} + s\frac{D_x f - D_y f}{2M}, \\ P_x(c) - s\frac{M}{2}\|x-c\|^2 &= P_y(c) + s\frac{M}{2}\|y-c\|^2, \\ D_x f + sM(x-c) &= D_y f - sM(y-c). \end{aligned} \tag{37}$$

Moreover, all minimal extensions of f coincide at c.

The proof of Lemma 7 uses [20], Propositions 2.2 and 2.13. The details are omitted. Throughout the remainder of this section, let c denote the point which satisfies Proposition 7.

Lemma 8 *Define* $\widetilde{P}_c \in \mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$ *as*

$$\widetilde{P}_c(z) \triangleq \tilde{f}_c + D_c\tilde{f}\cdot(z-c),\ \ z\in\mathbb{R}^d,$$

where

$$\tilde{f}_c \triangleq P_x(c) - s\frac{M}{2}\|x-c\|^2,$$

and

$$D_c\tilde{f} \triangleq D_x f + sM(x-c).$$

If $A(f;x,y)=0$*, then the following polynomial*

$$F(z) \triangleq \widetilde{P}_c(z) - s\frac{M}{2}\frac{[(z-c)\cdot(x-c)]^2}{\|x-c\|^2} + s\frac{M}{2}\frac{[(z-c)\cdot(y-c)]^2}{\|y-c\|^2},\ \ z\in\mathbb{R}^d$$

is a minimal extension of f.

If $A(f;x,y)\neq 0$*, let* $z\in\mathbb{R}^d$ *and set* $p(z)\triangleq(x-c)\cdot(z-c)$ *and* $q(z)\triangleq(y-c)\cdot(z-c)$*. We define*

$$F(z) \triangleq \begin{cases} \widetilde{P}_c(z) - s\frac{M}{2}\frac{[(z-c)\cdot(x-c)]^2}{\|x-c\|^2}, & \text{if } p(z)\geq 0 \text{ and } q(z)\leq 0,\\ \widetilde{P}_c(z) + s\frac{M}{2}\frac{[(z-c)\cdot(y-c)]^2}{\|y-c\|^2}, & \text{if } p(z)\leq 0 \text{ and } q(z)\geq 0,\\ \widetilde{P}_c(z), & \text{if } p(z)\leq 0 \text{ and } q(z)\leq 0,\\ \widetilde{P}_c(z) - s\frac{M}{2}\frac{[(z-c)\cdot(x-c)]^2}{\|x-c\|^2} + s\frac{M}{2}\frac{[(z-c)\cdot(y-c)]^2}{\|y-c\|^2}, & \text{if } p(z)\geq 0 \text{ and } q(z)\geq 0.\end{cases}$$

Then F *is a minimal extension of* f.

Remark 2 The function F is an extension of the 1-field f in the following sense. F defines a 1-field via its first order Taylor polynomials; in particular, define the 1-field U with $\mathrm{dom}(U)=\mathrm{dom}(F)$ as:

$$U(a)\triangleq J_aF,\ \ a\in\mathrm{dom}(F),$$

where J_aF is the first order Taylor polynomial of F. We then have:

$$U(x)=f(x) \quad \text{and} \quad U(y)=f(y),$$
$$\Gamma^1(U;\mathrm{dom}(U))=\Gamma^1(f;\mathrm{dom}(f)).$$

Proof After showing that the equality $A(f;x,y)=0$ implies that $(x-c)\cdot(c-y)=0$, the proof is easy to check. Suppose that $A(f;x,y)=0$. By (32) and (35) we have $M=B(f;x,y)$. By (37) we have

$$\|2c-(x+y)\| = \frac{\|D_xf-D_yf\|}{M} = \|x-y\|.$$

Therefore $(x-c)\cdot(c-y)=0$. $\square$

The proof of the following lemma is also easy to check.

Lemma 9 *Let* $g\in\mathscr{F}_{\Gamma^1}(\mathbb{X},Z)$ *such that for all* $a\in\mathrm{dom}(g)$, $g(a)=Q_a\in\mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$*, with* $Q_a(z)=g_a+D_ag\cdot(z-a)$*, where* $g_a\in\mathbb{R}$, $D_ag\in\mathbb{R}^d$*, and* $z\in\mathbb{R}^d$*. Suppose there exists* $P\in\mathscr{P}_2(\mathbb{R}^d,\mathbb{R})$ *such that*

$$P(a)=g_a,\ \ \nabla P(a)=D_ag, \text{ for all } a\in\mathrm{dom}(g).$$

Then

$$A(g;a,b)=0, \text{ for all } a,b\in\mathrm{dom}(g).$$

Proof Omitted.

Lemma 10 *All minimal extensions of f coincide on the line segments $[x,c]$ and $[c,y]$.*

Proof First, let F be the minimal extension of f defined in Lemma 8, and let U be the 1-field corresponding to F that was defined in remark 2. In particular, recall that we have:

$$U(a)(z) = J_aF(z) = F(a) + \nabla F(a)\cdot(z-a), \quad a \in \mathrm{dom}(F).$$

Now Let W be an arbitrary minimal extension of f such that for all $a \in \mathrm{dom}(W)$, $W(a) = Q_a \in \mathscr{P}^1(\mathbb{R}^d,\mathbb{R})$, with $Q_a(z) = W_a + D_aW\cdot(z-a)$, where $W_a \in \mathbb{R}$, $D_aW \in \mathbb{R}^d$, and $z \in \mathbb{R}^d$. We now restrict our attention to $[x,c]\cup[c,y]$. For any $a \in [x,c]\cup[c,y]$, we write $W(a) = Q_a$ in the following form:

$$Q_a(z) = F(a) + \nabla F(a)\cdot(z-a) + \delta_a + \Delta_a\cdot(z-a), \quad z \in \mathbb{R}^d,$$

where $\delta_a \in \mathbb{R}$ and $\Delta_a \in \mathbb{R}^d$. In particular, we have

$$W_a = F(a) + \delta_a,$$
$$D_aW = \nabla F(a) + \Delta_a.$$

Since U is a minimal extension of f, it is enough to show that $\delta_a = 0$ and $\Delta_a = 0$ for $a \in [x,c]\cup[c,y]$. By symmetry, without lost generality let us suppose that $a \in [x,c]$. Since W is a minimal extension of f, we have $W_x = F(x) = f_x$, and by Lemma 7, $W_c = F(c)$. Using (34) and (35), and once again since W is a minimal extension of f, the following inequality must be satisfied:

$$|A(W;e,a)| + \frac{B(W;e,a)^2}{2M} \le \frac{M}{2}, \quad e \in \{x,c\}. \tag{38}$$

Using Lemma 9 for U restricted to $\{x,a,c\}$ we have

$$A(U;e,a) = 0, \quad e \in \{x,c\}. \tag{39}$$

Therefore

$$A(W;e,a) = \frac{|-2\delta_a + \Delta_a\cdot(e-a)|}{\|e-a\|^2}, \quad e \in \{x,c\}. \tag{40}$$

Since $a \in [x,c]$, we can write $a = c + \alpha(x-c)$ with $\alpha \in [0,1]$. Using (38) and (39), the definition of U, and after simplification, δ_a and Δ_a must satisfy the following inequalities:

$$-2\delta_a + \alpha(1+s)\Delta_a\cdot(c-x) + \frac{\|\Delta_a\|^2}{2M} \le 0, \tag{41}$$

$$2\delta_a + \alpha(-1+s)\Delta_a\cdot(c-x) + \frac{\|\Delta_a\|^2}{2M} \le 0, \tag{42}$$

$$-2\delta_a - (1-\alpha)(1+s)\Delta_a\cdot(c-x) + \frac{\|\Delta_a\|^2}{2M} \le 0, \tag{43}$$

$$2\delta_a - (1-\alpha)(-1+s)\Delta_a\cdot(c-x) + \frac{\|\Delta_a\|^2}{2M} \le 0. \tag{44}$$

The inequality $(1-\alpha)((41)+(42)) + \alpha((43)+(44))$ implies that $\Delta_a = 0$. Furthermore, the inequalities (41) and (42) imply that $\delta_a = 0$. Now the proof is complete. □

We finish this appendix by proving Proposition 2. Let us use the notations of Proposition 2 where c satisfies Lemma 7. By Lemma 10, the extension U (defined in Remark 2) of f is the unique minimal extension of f on the restriction to $[x,c]\cup[c,y]$. Moreover, we can check that

$$\Gamma^1(f;x,y) = \max\{\Gamma^1(U;x,a), \Gamma^1(U;a,y)\}, \text{ for all } a \in [x,c]\cup[c,y]. \tag{45}$$

Let W be an extension of f. By contradiction suppose that there exists $a \in [x,c]\cup[c,y]$ such that

$$\Gamma^1(f;x,y) > \max\{\Gamma^1(W;x,a), \Gamma^1(W;a,y)\}. \tag{46}$$

Using [20], Theorem 2.6, for the 1-field $g \triangleq \{f(x), W(a), f(y)\}$ of domain $\{x,a,y\}$ there exists an extension G of g such that

$$\Gamma^1(G;\mathrm{dom}(G)) \le \Gamma^1(f;x,y). \tag{47}$$

Therefore G is a minimal extension of f. By Lemma (10) and the definition of G we have $W(a) = G(a) = U(a)$. But then by (45),(46), and (47) we obtain a contradiction. Now the proof of the Proposition 2 is complete. □

A.4 (P_4) for Γ^1

Property (P_4) (continuity of Γ^1) can be shown using (34), and a series of elementary calculations. We omit the details.

A.5 (P_5) for Γ^1

To show property (P_5) (continuity of $f \in \mathscr{F}_{\Gamma^1}(\mathbb{X}, Z)$), we first recall the definition of d_Z. For $P \in \mathscr{P}^1(\mathbb{R}^d, \mathbb{R})$ with $P(a) = p_0 + D_0 p \cdot a$, $p_0 \in \mathbb{R}$, $D_0 p \in \mathbb{R}^d$, we have

$$d_Z(P, Q) = |p_0 - q_0| + \|D_0 p - D_0 q\|.$$

Recall also that for a 1-field $f : E \to Z$, $E \subset \mathbb{X}$, we have:

$$x \in E \mapsto f(x)(a) = f_x + D_x f \cdot (a - x) = (f_x - D_x f \cdot x) + D_x f \cdot a, \quad a \in \mathbb{R}^d.$$

To show continuity of $f \in \mathscr{F}_{\Gamma^1}(\mathbb{X}, Z)$ at $x \in \mathbb{X}$, we need the following: for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|x - y\| < \delta$, then $d_Z(f(x), f(y)) < \varepsilon$. Consider the following:

$$\begin{aligned} d_Z(f(x), f(y)) &= |f_x - D_x f \cdot x - f_y + D_y f \cdot y| + \|D_x f - D_y f\| \\ &\leq |f_x - f_y| + |D_x f \cdot x - D_y f \cdot y| + \|D_x f - D_y f\|. \end{aligned} \tag{48}$$

We handle the three terms (48) separately and in reverse order.

For the third term, recall the definition of $B(f; x, y)$ in (33), and define $B(f; E)$ accordingly; we then have:

$$\|D_x f - D_y f\| \leq B(f; E)\|x - y\| \leq \Gamma^1(f; E)\|x - y\|. \tag{49}$$

Since $\Gamma^1(f; E) < \infty$, that completes this term.

For the second term:

$$\begin{aligned} |D_x f \cdot x - D_y f \cdot y| &\leq |D_x f \cdot (x - y)| + |(D_x f - D_y f) \cdot y| \\ &\leq \|D_x f\|\|x - y\| + \|D_x f - D_y f\|\|y\| \end{aligned}$$

Using (49), we see that this term can be made arbitrarily small using $\|x - y\|$ as well.

For the first term $|f_x - f_y|$, define $g : E \to \mathbb{R}$ as $g(x) = f_x$ for all $x \in E$. By Proposition 2.5 of [20], the function g is continuous. This completes the proof. □

B Proof of Proposition 1

We prove Proposition 1, which we restate here:

Proposition 3 (Proposition 1) *Let $f \in \mathscr{F}_\Phi(\mathbb{X}, Z)$. For any open $V \subset\subset \operatorname{dom}(f)$ and $\alpha \geq 0$, let us define*

$$V_\alpha \triangleq \{x \in V \mid d_{\mathbb{X}}(x, \partial V) \geq \alpha\}.$$

Then for all $\alpha > 0$ and for all open $V \subset\subset \operatorname{dom}(f)$,

$$\Phi(f; V_\alpha) \leq \max\{\Psi(f; V; \alpha), \Phi(u; \partial V_\alpha)\}, \tag{50}$$

and

$$\Phi(f; V) = \max\{\Psi(f; V; 0), \Phi(f; \partial V)\}. \tag{51}$$

Proof For the first statement fix $\alpha > 0$ and an open set $V \subset\subset \mathbb{X}$. For proving (50), it is sufficient to prove that for all $x \in \mathring{V}_\alpha$ and for all $y \in V_\alpha$ we have

$$\Phi(f;x,y) \leq \max\{\Psi(f;V;\alpha), \Phi(f;\partial V_\alpha)\}. \tag{52}$$

Fix $x \in \mathring{V}_\alpha$. Let $B(x;r_x) \subset V$ be a ball such that r_x is maximized and define

$$M(x) \triangleq \sup\left\{\Phi(f;x,y) \mid y \in \overline{V_\alpha \setminus B(x;r_x)}\right\},$$

as well as

$$\Delta(x) \triangleq \left\{y \in \overline{V_\alpha \setminus B(x;r_x)} \mid \Phi(f;x,y) = M(x)\right\},$$

and

$$\delta(x) \triangleq \inf\{d_{\mathbb{X}}(x,y) \mid y \in \Delta(x)\}.$$

We have three cases:

Case 1 Suppose $M(x) \leq \sup\{\Phi(f;x,y) \mid y \in B(x;r_x)\}$. Since $B(x;r_x) \subset V$ with $r_x \geq \alpha$ we have

$$\Phi(f;x,y) \leq \Psi(f;V;\alpha), \quad \forall\, y \in B(x;r_x).$$

Therefore $M(x) \leq \Psi(f;V;\alpha)$. That completes the first case.

For cases two and three, assume that $M(x) > \sup\{\Phi(f;x,y) \mid y \in B(x;r_x)\}$ and select $y \in \Delta(x)$ with $d_{\mathbb{X}}(x,y) = \delta(x)$.

Case 2 Suppose $y \in \operatorname{int}(V_\alpha \setminus B(x;r_x))$. Let $B(y;r_y) \subset V$ be a ball such that r_y is maximal. Consider the curve $\gamma \in \Gamma(x,y)$ satisfying (P_3). Let $m \in \gamma \cap B(y;r_y) \cap V_\alpha$, $m \neq x,y$. Using (P_3), we have

$$\Phi(f;x,y) \leq \max\{\Phi(f;x,m), \Phi(f;m,y)\}. \tag{53}$$

Using the monotonicity of γ we have $d_{\mathbb{X}}(x,m) < d_{\mathbb{X}}(x,y)$. Using the minimality of the distance of $d_{\mathbb{X}}(x,y)$ and since $m \in V_\alpha$ we have $\Phi(f;x,m) < \Phi(f;x,y)$. Therefore

$$\Phi(f;x,y) \leq \Phi(f;m,y). \tag{54}$$

Since $m \in B(y;r_y)$ with $r_y \geq \alpha$, using the definition of Ψ we have $\Phi(f;m,y) \leq \Psi(f;V;\alpha)$. Therefore $M(x) \leq \Psi(f;V;\alpha)$.

Case 3 Suppose $y \in \partial V_\alpha \setminus B(x;r_x)$. As in case two, let $B(y;r_y) \subset V$ be a ball such that r_y is maximal and consider the curve $\gamma \in \Gamma(x,y)$ satisfying (P_3). Let $m \in \gamma \cap B(y;r_y) \cap V_\alpha$. If there exists $m \neq y$ in V_α, we can apply the same reasoning as in case two and we have $M(x) \leq \Psi(f;V;\alpha)$.

If $m = y$ is the only element of $\gamma \cap B(y;r_y) \cap V_\alpha$, then there still exists $m' \in \gamma \cap \partial V_\alpha$ with $m' \neq y$. Using (P_3) we have

$$\Phi(f;x,y) \leq \max\{\Phi(f;x,m'), \Phi(f;m',y)\}. \tag{55}$$

Using the monotonicity of γ we have $d_{\mathbb{X}}(x,m') < d_{\mathbb{X}}(x,y)$. Using the minimality of distance of $d_{\mathbb{X}}(x,y)$ and since $m' \in V_\alpha$ we have $\Phi(f;x,m') < \Phi(f;x,y)$. Therefore

$$\Phi(f;x,y) \leq \Phi(f;m',y). \tag{56}$$

Since $m', y \in \partial V_\alpha$, we obtain the following majoration

$$\Phi(f;x,y) \leq \Phi(f;m',y) \leq \Phi(f;\partial V_\alpha), \tag{57}$$

which in turn gives:

$$M(x) \leq \Phi(f;\partial V_\alpha).$$

The inequality (50) is thus demonstrated.

For the second statement, we note that by the definition of Ψ we have

$$\max\{\Psi(f;V;0),\Phi(f;\partial V)\} \leq \Phi(f;V).$$

Using (50), to show (51) it is sufficient to prove

$$\lim_{\alpha\to 0} \Phi(f;V_\alpha) = \Phi(f;V). \tag{58}$$

Let $\varepsilon > 0$. Then there exists $x_\varepsilon \in V$ and $y_\varepsilon \in \overline{V}$ such that

$$\Phi(f;V) \leq \Phi(f;x_\varepsilon,y_\varepsilon)+\varepsilon.$$

Set $r_\varepsilon = d_{\mathbb{X}}(x_\varepsilon,\partial V)$. If $y_\varepsilon \in V$, there exists τ_1 with $0<\tau_1 \leq r_\varepsilon$ such that for all α, $0<\alpha\leq\tau_1$, $(x_\varepsilon,y_\varepsilon) \in V_\alpha \times V_\alpha$. Therefore

$$\Phi(f;x_\varepsilon,y_\varepsilon) \leq \Phi(f;V_\alpha), \quad \forall\alpha,\ 0<\alpha\leq\tau_1.$$

If, on the other hand, $y_\varepsilon \in \partial V$, using (P_4) there exists τ_2 with $0<\tau_2 \leq \min\{r_\varepsilon,\tau_1\}$, such that

$$|\Phi(f;x_\varepsilon,m)-\Phi(f;x_\varepsilon,y_\varepsilon)| \leq \varepsilon, \quad \forall m \in B(y_\varepsilon;\tau_2).$$

By choosing $m \in B(y_\varepsilon;\tau_2)\cap V_{\tau_2}$, we obtain

$$\Phi(f;x_\varepsilon,y_\varepsilon) \leq \Phi(f;V_\alpha)+\varepsilon, \quad \forall\alpha,\ 0<\alpha\leq\tau_2.$$

Therefore $\Phi(f;V) \leq \Phi(f;V_\alpha)+2\varepsilon$, for all α such that $0<\alpha\leq\tau_2$ and for all $\varepsilon>0$. Thus (58) is true. □

References

1. Scott N. Armstrong and Charles K. Smart. As easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions. *Calculus of Variations and Partial Differential Equations*, 37(3-4):381–384, 2010.
2. Gunnar Aronsson. Minimization problems for the functional $\sup_x f(x,f(x),f'(x))$. *Arkiv för Matematik*, 6:33–53, 1965.
3. Gunnar Aronsson. Minimization problems for the functional $\sup_x f(x,f(x),f'(x))$ II. *Arkiv för Matematik*, 6:409–431, 1966.
4. Gunnar Aronsson. Extension of functions satisfying Lipschitz conditions. *Arkiv för Matematik*, 6:551–561, 1967.
5. Gunnar Aronsson, Michael G. Crandall, and Petri Juutinen. A tour of the theory of absolutely minimizing functions. *Bulletin of the American Mathematical Society*, 41:439–505, 2004.
6. Guy Barles and Jérôme Busca. Existence and comparison results for fully nonlinear degenerate elliptic equations. *Communications in Partial Differential Equations*, 26(11–12):2323–2337, 2001.
7. Robert Jensen. Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient. *Archive for Rational Mechanics and Analysis*, 123(1):51–74, 1993.
8. Petri Juutinen. Absolutely minimizing Lipschitz extensions on a metric space. *Annales Academiae Scientiarum Fennicae Mathematica*, 27:57–67, 2002.
9. John L. Kelley. Banach spaces with the extension property. *Transactions of the American Mathematical Society*, 72(2):323–326, 1952.
10. Mojżesz David Kirszbraun. Über die zusammenziehende und Lipschitzsche Transformationen. *Fundamenta Mathematicae*, 22:77–108, 1934.
11. Edward James McShane. Extension of range of functions. *Bulletin of the American Mathematical Society*, 40(12):837–842, 1934.
12. Earl J. Mickle. On the extension of a transformation. *Bulletin of the American Mathematical Society*, 55(2):160–164, 1949.
13. Victor A. Milman. Absolutely minimal extensions of functions on metric spaces. *Matematicheskii Sbornik*, 190(6):83–110, 1999.
14. Leopoldo Nachbin. A theorem of the Hahn-Banach type for linear transformations. *Transactions of the American Mathematical Society*, 68(1):28–46, 1950.

15. Assaf Naor and Scott Sheffield. Absolutely minimal Lipschitz extension of tree-valued mappings. *Mathematische Annalen*, 354(3):1049–1078, 2012.
16. Yuval Peres, Oded Schramm, Scott Sheffield, and David B. Wilson. Tug-of-war and the infinity Laplacian. *Journal of the American Mathematical Society*, 22(1):167–210, 2009.
17. Isaac Schoenberg. On a theorem of Kirszbraun and Valentine. *American Mathematical Monthly*, 60:620–622, 1953.
18. Scott Sheffield and Charles K. Smart. Vector-valued optimal Lipschitz extensions. *Communications on Pure and Applied Mathematics*, 65(1):128–154, 2012.
19. Erwan Le Gruyer. On absolutely minimizing Lipschitz extensions and PDE $\Delta_\infty(u) = 0$. *NoDEA: Nonlinear Differential Equations and Applications*, 14(1-2):29–55, 2007.
20. Erwan Le Gruyer. Minimal Lipschitz extensions to differentiable functions defined on a Hilbert space. *Geometric and Functional Analysis*, 19(4):1101–1118, 2009.
21. Frederick Albert Valentine. A Lipschitz condition preserving extension for a vector function. *American Journal of Mathematics*, 67(1):83–93, 1945.
22. James H. Wells and Lynn R. Williams. *Embeddings and Extensions in Analysis*. Springer-Verlag, New York Heidelberg Berlin, 1975.
23. John C. Wells. Differentiable functions on Banach spaces with Lipschitz derivatives. *Journal of Differential Geometry*, 8:135–152, 1973.
24. Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets. *Transactions of the American Mathematical Society*, 36(1):63–89, 1934.